Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 4, 2002*
Revised: *August 1, 2002*
Accepted: *September 2, 2002*

An approach to $\mathcal{N}=4$ ADE gauge theory on $K3$

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Abstract: We propose a recipe for determination of the partition function of $\mathcal{N}=4$ ADE gauge theory on $K3$ by generalizing our previous results of the SU(N) case. The resulting partition function satisfies Montonen-Olive duality for ADE gauge group.

Keywords: Conformal Field Models in String Theory, Gauge Symmetry, Supersymmetry and Duality, Differential and Algebraic Geometry.

Contents

1. Introduction

This article is the continuation of our previous works [6, 7]. Based on the understanding obtained in these works, we try to determine the partition function of $\mathcal{N}=4$ ADE gauge theory on $K3$.

$\mathcal{N}=4$ theory is one of the good laboratories of testing duality conjecture. It is well-known that $\mathcal{N}=4$ theory has the largest possible number of super-symmetry for a four-dimensional theory without gravity. Furthermore, it is believed to be exactly finite and conformally invariant. The duality of this theory is believed to be the one proposed by Montonen and Olive. On the basis of this thought, the study of $\mathcal{N}=4$ SU(N) gauge theory has enjoyed many progresses with the aid of the S-duality conjecture: the sharpened version of Montonen-Olive duality [21, 11, 13]. On the other hand, there has been almost no attempt to determine the partition function of D, E gauge theory in spite of the fact that the S duality conjecture for these cases has already been established [22, 16].

In our previous work, we have studied $\mathcal{N}=4$ super-symmetric Yang-Mills theory on orbifold-$T^4/\mathbb{Z}_2$ for SU(2) [6] and for SU(N) [7]. In these works, we tried to realize the following geometrical picture at the partition function level. A special class of $K3$ surfaces can be constructed from blowing up sixteen singularities of the quotient space $T^4/\mathbb{Z}_2 = S_0$. Therefore, the partition function on $K3$ should be given as the product of the partition function of S_0 and the blow-up formula [24, 21, 9]. This attempt was successful not only for SU(2) but also for SU(N). The verification of this attempt mainly depends on the following two facts. The first one is that the SU(N) partition function on $K3$ can be described by a Hecke transformation of order N of $1/\eta(\tau)^{24}$ [21, 13, 26, 11]. The second one is that $1/\eta(\tau/N)$ can be written as the form $\theta^2_{A_{N-1}}(\tau)/\eta(\tau)^N$, where $\theta^2_{A_{N-1}}(\tau)$ is a kind of A_{N-1} theta function [7]. This identity is easily verified from the celebrated denominator identity of the affine Lie algebra [8, 12]. These two facts enabled us to rewrite the partition function of SU(N) theory on $K3$ into the desired form.

Looking back at the above verification, we can speculate that inversion of the above process can be used to guess the form of the partition function of general ADE gauge theory. Fortunately, we already have the works on ADE blow-up formulas [24, 19, 9], which states that principal part of ADE blow-up formulas have the form of ADE theta functions $\theta_{A,D,E}(\tau)$. In this paper, we proceed further and assert that the partition function of ADE gauge theory should take the form of $(\theta_{ADE}(\tau)/\eta(\tau)^{r+1})^{24}$ (r is the rank of the gauge group). This form can be verified by using the fact that $\mathcal{N}=4$ ADE gauge theory on $K3$ is obtained from type-IIA string theory compactified on $K3 \times T^2 \times (ALE$ space). By plugging this assertion into the denominator identity, we can see that the D, E partition functions can be rewritten as the form of $(\eta - \text{product})^{24}$. As was pointed out before, the partition function of SU(N) gauge theory on $K3$ surface is written only by η function. This is because the moduli space of rank N semi-stable sheaves is described by Hilbert scheme of $K3$ surface itself [17, 18, 26]. Physical interpretation of this fact is also given from the point of view of type-IIA–heterotic duality. Though we have not obtain analogous results on the moduli space of D, E vector bundles, we already have many beautiful results on $K3$ surface, which is enough for us to expect that the partition function of D, E gauge theory should be written only by η-product. Therefore, we use the above (η-product)^{24} as the primary building block of the D, E partition function on $K3$. Next step of our construction is found from the analogy with the A_{N-1} case. The corresponding primary building block of the A_{N-1} partition function is $1/\eta(\tau/N)^{24}$. Then we take $\mathrm{SL}(2,\mathbb{Z})$ orbit of this function (precisely speaking, we have to include the transformation $\tau \to \tau + 1/m$ in some cases). After that, we take linear combination of all the building blocks so obtained, whose coefficients are determined to satisfy the S-duality conjecture, as was done in Vafa-Witten 's original work [21]. The result is nothing but the order N Hecke transformation of $1/\eta(\tau)^{24}$. Hence we applied the same procedure by taking (η-product)^{24} as the primary function. As a result, we found that the above procedure also works quite well in this case. Remarkably enough, the resulting partition function completely satisfies Montonen-Olive duality! [16].

Unfortunately, our results are not complete from the geometrical point of view. It is well-known that the partition function is the generating function of the Euler number of the moduli space of irreducible anti-self-dual connection. For ADE gauge group, we have Atiyah-Hitchin-Singer formula that gives us the dimension of the moduli space of irreducible anti-self-dual connection [1]. This formula tells us that if the instanton number k is less than $\mathrm{rank}(\mathcal{G}_{ADE}) + 1$, the moduli space is empty. On the other hand, the partition function that we have derived has non-vanishing coefficients in the region $k \leq r$. Up to now, we think that these phenomena is caused by existence of reducible connections. Especially in the case of $\mathrm{SU}(N)$ theory on the manifold $b_2^+ \leq 1$, it is well-known that existence of reducible connections gives rise to the appearance of the holomorphic anomaly in the partition function. Thus we speculate that the true partition function of D, E theory on $K3$ should have the holomorphic anomaly. In other words, we have derived the holomorphic part of the partition function of D, E gauge theory. Therefore, we should search for an analogue of the holomorphic anomaly equation used in [13] as the second step. This problem will be discussed in our subsequent paper.

The organization of this paper is the following: in section 2 we review the Vafa-Witten theory in the $\mathrm{SU}(N)$ case and discuss generalization to the D, E gauge theory. Next, we look carefully at the detailed structure of the duality conjecture on the partition function of $\mathrm{SU}(N)$ theory as warming-ups. In section 3 we estimate the form of the D, E partition functions on $K3$ from the point of view of type-IIA string theory compactified on $K3 \times T^2 \times (ALE \text{ space})$. In section 4 we combine the above result with the denominator identity of affine Lie algebra and determine the D, E partition function on $K3$ up to some small rank. In section 5 we point out that our partition function of D, E gauge theory on $K3$ does not satisfy the gap condition and discuss the possible improvement parallel to the treatment of holomorphic anomaly in the $\frac{1}{2}K3$ case. In section 6 we conclude and discuss the remaining problems.

2. Vafa-Witten theory

2.1 't Hooft flux for ADE gauge theory

In this subsection, we introduce the 't Hooft flux for ADE gauge theory. To this aim, we first consider the non-abelian version of the Dirac quantization condition [16]:

$$\exp(4\pi i e F_0) = 1\,, \tag{2.1}$$

where F_0 is a Lie algebra valued matrix of ADE gauge group $\mathcal{G}$ and appears in the solution of the magnetic monopole equation. By using gauge transformation, F_0 can be rewritten into the form:

$$eF_0 = \sum_{j=1}^{\mathrm{rank}(\mathcal{G})} \beta_j T_j\,, \tag{2.2}$$

where the T_j's are any appropriately normalized set of mutually commutative generators of $\mathcal{G}$. Here we introduce the root lattice $\Lambda(\mathcal{G})$. Then (2.1) turns out to be

$$2\beta \cdot \alpha \in \mathbb{Z}\,, \qquad \text{for all } \alpha \in \Lambda(\mathcal{G})\,. \tag{2.3}$$

This condition tells us that β should belong to the dual lattice (or weight lattice) $\Lambda(\mathcal{G}^\vee)$. On the other hand, it is well-known that

$$\Lambda(\mathcal{G}^\vee)/\Lambda(\mathcal{G}) \cong \Gamma_{\mathcal{G}}\,, \tag{2.4}$$

where $\Gamma_{\mathcal{G}}$ is center of $\mathcal{G}$. In this way, Montonen and Olive proposed that ADE gauge group $\mathcal{G}$ is dual to $\mathcal{G}/\Gamma_{\mathcal{G}}$. From this proposal, it is natural to assume that 't Hooft flux of the dual group $\mathcal{G}/\Gamma_{\mathcal{G}}$ is given by $v \in H^2(X, \Gamma_{\mathcal{G}})$. For later use, we write down $\Gamma_{\mathcal{G}}$ in table 1. Classification of the type of the 't Hooft flux of A_{N-1} (N: prime) is already studied [21, 11]. However, general N case is more complicated. In section 2.3, we will show explicitly how $v \in H^2(K3, \Gamma_{\mathcal{G}})$ are classified in the case of general ADE gauge group.

$\mathcal{G}$	$\Gamma_{\mathcal{G}}$
A_{N-1}	$\mathbb{Z}_N$
D_{2N}	$\mathbb{Z}_2 \times \mathbb{Z}_2$
D_{2N+1}	$\mathbb{Z}_4$
E_6	$\mathbb{Z}_3$
E_7	$\mathbb{Z}_2$
E_8	trivial

Table 1: Center of $\mathcal{G}$.

2.2 Vafa-Witten conjecture for ADE gauge theory

Following [21, 11, 25], we review the general structure of Vafa-Witten conjecture. Twisted $\mathcal{N} = 4$ gauge theory on the manifold with suitable vanishing theorem causes remarkable simplification [21]. That is, its partition function has the form of the summation of the Euler number of the moduli space of the ASD equations. More precisely, for twisted $\mathcal{N} = 4$ $\mathcal{G}/\Gamma_{\mathcal{G}}$ gauge theory with 't Hooft flux $v \in H^2(X, \Gamma_{\mathcal{G}})$ on X, we propose that the partition function of this theory is given by the formula:

$$Z_v^X(\tau) := q^{-\frac{(r+1)\chi(X)}{24}} \sum_k \chi(\mathcal{M}(v,k)) q^k \quad (q := \exp(2\pi i \tau))\,, \tag{2.5}$$

where $\mathcal{M}(v,k)$ is the moduli space of ASD connections associated to $\mathcal{G}/\Gamma_{\mathcal{G}}$-principal bundle with 't Hooft flux v and fractional instanton number $k \in \frac{1}{2|\Gamma_{\mathcal{G}}|}\mathbb{Z}$. In (2.5), τ is the gauge coupling constant including theta angle, $\chi(X)$ is Euler number of X and r is the rank of $\mathcal{G}$. Introduction of $q^{-1/24}$ factor is required by the modular property like the case of $\eta(\tau)$ function. For $\mathrm{SU}(N)/\mathbb{Z}_N$, this conjecture is nothing but Vafa-Witten's [21]. With this result, Vafa and Witten conjectured the behavior of the partition functions under the action of $\mathrm{SL}(2,\mathbb{Z})$ on τ. They started with 't Hooft's work [20] in mind. In [20], the path integral with $\mathbb{Z}_N$-valued electric flux and that with magnetic flux is related by Fourier transform. Vafa and Witten combined the conjecture of strong/week duality to this 't Hooft's result. Their conjecture is summarized by the following formula:

$$Z_v^X\left(-\frac{1}{\tau}\right) = |\Gamma_{\mathcal{G}}|^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} \cdot \sum_{u\in H^2(X,\Gamma_{\mathcal{G}})} \zeta_{\mathcal{G}}^{u\cdot v} Z_u^X(\tau)\,, \tag{2.6}$$

where $\zeta_{\mathcal{G}} = \exp(2\pi i/N_{\mathcal{G}})$. We conjecture that this duality relation is also valid for the general ADE partition function (2.5).

For later use, we introduce the notation:

$$\begin{aligned} Z_{\mathcal{G}}^X(\tau) &:= \frac{1}{|\Gamma_{\mathcal{G}}|} Z_t^X(\tau)\,, \\ Z_{\mathcal{G}/\Gamma_{\mathcal{G}}}^X(\tau) &:= \sum_{u\in H^2(X,\Gamma_{\mathcal{G}})} Z_u^X(\tau)\,. \end{aligned} \tag{2.7}$$

In this notation, we can obtain the following formula from (2.6):

$$Z_t^X\left(-\frac{1}{\tau}\right) = |\Gamma_{\mathcal{G}}|^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} Z_{\mathcal{G}/\Gamma_{\mathcal{G}}}^X(\tau)\,. \tag{2.8}$$

This formula is one of the key points on their explicit determination of the form of the partition function of complex surface with ample canonical bundle.

2.3 Counting the number of orbits on $K3$

In the rest of this paper, we will concentrate on ADE gauge theory on $K3$. In [21], Vafa and Witten classified the types of $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ theory with 't Hooft flux $v \in H^2(K3,\mathbb{Z}_2)$ on $K3$. Note that $b^2(K3) = 22$, and v can takes 2^{22} values. They used the property that diffeomorphism invariant of v is given by v^2 mod 4 ($v \neq 0$) and $v = 0$. Especially on $K3$ there are only three types of partition functions. These three types are called as even ($v^2 \equiv 0$ mod 4 and $v \neq 0$), odd ($v^2 \equiv 2$ mod 4) and trivial ($v = 0$) respectively. Furthermore numbers of each type are denoted as $n_{\mathrm{even}}, n_{\mathrm{odd}}$ and n_0 respectively. These numbers are concretely counted in [21]. Of course they sum up to $n_0 + n_{\mathrm{even}} + n_{\mathrm{odd}} = 2^{22}$. It is well known that the intersection form on $H^2(K3,\mathbb{Z})$ is $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. However Vafa and Witten took $H^{\oplus 11}$ (the direct sum of 11 copies of H) instead of $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ itself as the intersection form on $H^2(K3,\mathbb{Z})$ for convenience. Here

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.9}$$

This replacement makes counting the numbers of orbits very simple. We follow this replacement in this paper. From now on, we generalize the above result of the SU(2) case to the case of general ADE group. First we count naive number of orbits for $\mathrm{SU}(N)/\mathbb{Z}_N$. We denote 't Hooft flux for $\mathrm{SU}(N)/\mathbb{Z}_N$ by,

$$v_N \in (H^{\oplus 11}) \otimes_{\mathbb{Z}} \mathbb{Z}_N =: H_N\,. \tag{2.10}$$

Roughly speaking, the word "naive" means that v_N has no restriction. Explicitly, the naive number of orbits of $\mathrm{SU}(N)/\mathbb{Z}_N$ is defined as follows:

$$n_j(N) = \text{number of } v_N^2 \equiv 2j \mod 2N\,. \tag{2.11}$$

We introduce V whose intersection form is given by H. For V, we define,

$$l_j = \text{numbers of } V^2 \equiv 2j \mod 2N\,. \tag{2.12}$$

With this definition, one can build the recursive formula for computing the number of orbits on the manifold with $H^{\oplus n}$. The solution for $n = 11$ case is

$$n_j(N) = \frac{1}{N}\sum_{k=0}^{N-1} m_k^{11}\zeta_N^{jk}\,, \tag{2.13}$$

where

$$m_k = \sum_{j=0}^{N-1} l_j\zeta_N^{jk}\,, \tag{2.14}$$

and

$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right).$$

From now on, we introduce a subscript "t" such as n_t, which corresponds to the trivial flux $v = 0$.

A_{N-1}: N prime. In this part, we count the number of orbits of $\mathrm{SU}(N)/\mathbb{Z}_N$:$N$ prime. If N is prime number, one finds $m_0 = 2N-1, m_j = N, (j = 1,\ldots,N-1)$. With some algebra, one can find

$$\begin{aligned}
n_0^N &= \text{number of } v_N^2 \equiv 0 \mod 2N \text{ and } v_N \neq 0\\
&= n_0(N) - 1 = N^{21} + (N-1)N^{10} - 1\,, && (2.15)\\
n_j^N &= \text{number of } v_N^2 \equiv 2j \mod 2N\\
&= n_j(N) = N^{21} - N^{10}, j = 1,\ldots,N-1\,, && (2.16)\\
n_t^1 &= \text{number of } v_N = 0\\
&= n_t = 1\,. && (2.17)
\end{aligned}$$

A_{N-1}: N non-prime. In this part, we count the number of orbits of $\mathrm{SU}(N)/\mathbb{Z}_N$:$N$ non-prime. That is $N = p_1^{m_1}p_2^{m_2}\cdots p_l^{m_l}$ case (p_j is a prime). Here we took the following two cases as typical examples: (i) $N = p^m$, (ii) $N = pq$. Other cases are straightforward generalization of these two cases. Before moving to specific discussion, we explain general structure of non-prime N case. For non-prime N, there are several blocks of type, which are classified by all possible divisor of $N = p_1^{m_1}p_2^{m_2}\cdots p_l^{m_l}$. That is

$$p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}, 0 \le M_1 \le m_1, 0 \le M_2 \le m_2, \ldots, 0 \le M_l \le m_l\,.$$

Each block labeled by $p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}$ has the types j $(0 \le j \le p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}-1)$. Thus we denote the number of orbits of block m and type j by n_j^m. Now, we give explicit definitions of these materials. First, we define $C_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} \subset H_N$ as follows.

$$C_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} := \left\{ v_N \in H_N \mid p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l} v_N = 0 \right\}. \tag{2.18}$$

Notice that $C_m \subset C_n$ if and only if $m|n$, and we can easily see the following equality:

$$C_{i_1} \cap C_{i_2} \cap C_{i_3} \cap \cdots \cap C_{i_j} = C_{\gcd(i_1,i_2,\cdots,i_j)}\,. \tag{2.19}$$

Since the condition $p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l} v_N = 0$ means that all the components of v_N must be multipliers of $p_1^{m_1-M_1}p_2^{m_2-M_2}\cdots p_l^{m_l-M_l}$, we can construct a natural isomorphism:

$$\begin{aligned}
&\phi_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} : C_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} \to H_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}\,, \\
&\phi(v_N) := \frac{1}{p_1^{m_1-M_1}p_2^{m_2-M_2}\cdots p_l^{m_l-M_l}} v_N \in H_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}\,.
\end{aligned} \tag{2.20}$$

With these set-up, we define the block $B_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}$ as follows:

$$B_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} := C_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}} \setminus \left(\bigcup_{\substack{m|p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l},\\ m\neq p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}} (C_m) \right). \tag{2.21}$$

Then the number $n_j^{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}$ is nothing but the number of $v_N \in B_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}$ that satisfies the condition:

$$(\phi_{p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}}(v_N))^2 \equiv 2j \mod 2p_1^{M_1}p_2^{M_2}\cdots p_l^{M_l}\,. \tag{2.22}$$

(i) $N = p^m$ case. In this case, we have m blocks $B_{p^M} = C_{p^M} \setminus C_{p^{M-1}}$ $(M = 0,1,2,\ldots,m)$ and each block B_{p^M} has p^M types. Notice that we have the following inclusion sequence in this case:

$$\{0\} = C_1 \subset C_p \subset C_{p^2} \subset \cdots \subset C_{p^{m-1}} \subset C_{p^m} = H_{p^m}\,. \tag{2.23}$$

Since we have the isomorphism $\phi_{p^M} : C_{p^M} \to H_{p^M}$, we can write down $n_j^{p^M}$ in terms of naive numbers $n_i(p^M)$ and $n_l(p^{M-1})$. The result is the following:

- Block B_{p^M}, $(M = m, m-1, \ldots, 1)$

$$n_{kp^2}^{p^M} = n_{kp^2}(p^M) - \sum_{l=0}^{p-1} n_{lp^{M-2}+k}(p^{M-1})\,, \qquad k = 0,\ldots,p^{M-2}-1\,, \tag{2.24}$$

$$n_j^{p^M} = n_j(p^M)\,, \qquad j = 1,\ldots,p^M-1 \quad \text{and} \quad j \neq kp^{M-2}\,, \tag{2.25}$$

- Block B_1,

$$n_t^1 = n_t = 1\,. \tag{2.26}$$

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For later use, we explicitly write down the results in the $N = 4$ case.

$$
\begin{aligned}
n_0^4 &= n_0(4) - n_0(2) - n_1(2) = 2^{42} + 2^{31} - 2^{21}\,,\\
n_1^4 &= n_1(4) = 2^{42} - 2^{31}\,,\\
n_2^4 &= n_2(4) = 2^{42} + 2^{31} - 2^{21}\,,\\
n_3^4 &= n_3(4) = 2^{42} - 2^{31}\,,\\
n_0^2 &= n_0(2) - n_t = 2^{21} + 2^{10} - 1\,,\\
n_1^2 &= n_1(2) = 2^{21} - 2^{10}\,,\\
n_t^1 &= n_t = 1\,.
\end{aligned}
\tag{2.27}
$$

(ii) $N = pq$ **case.** In this case, we have 4 blocks:

$$
B_{pq} = C_{pq}\backslash(C_p \cup C_q)\,,\quad B_p = C_p\backslash C_1\,,\quad B_q = C_q\backslash C_1\,,\quad B_1 = C_1 = \{0\}\,,\quad (C_p \cap C_q = C_1)\,.
\tag{2.28}
$$

Therefore, we can evaluate n_j^m by use of naive numbers in the same way as the previous case.

- Block B_{pq},

$$
\begin{aligned}
n_0^{pq} &= n_0(pq) - n_0(q) - n_0(p) + n_t\,, && && (2.29)\\
n_{kp}^{pq} &= n_{kp}(pq) - n_{l_k^q}(q)\,, && k = 0, \ldots, q-1\,, && (2.30)\\
n_{kq}^{pq} &= n_{kq}(pq) - n_{l_k^p}(p)\,, && k = 0, \ldots, p-1\,, && (2.31)\\
n_j^{pq} &= n_j(pq)\,, && j = 1, \ldots, pq-1\,,\quad j \neq kp\,,\quad j \neq kq\,, && (2.32)
\end{aligned}
$$

  where l_k^q and l_k^p satisfy

$$
l_k^q p^2 \equiv 2kp \bmod 2pq\,, \qquad l_k^p q^2 \equiv 2kq \bmod 2pq\,.
\tag{2.33}
$$

- Block B_p,

$$
\begin{aligned}
n_0^p &= n_0(p) - n_t\,, && && (2.34)\\
n_j^p &= n_j(p)\,, && j = 1, \ldots, q-1\,, && (2.35)
\end{aligned}
$$

- Block B_q, numbers are given by $p \leftrightarrow q$.

- Block B_1,

$$
n_t^1 = n_t = 1\,.
\tag{2.36}
$$

In the rest of this subsection, we count the number of orbits of $\mathcal{G}/\Gamma_{\mathcal{G}}$. However the counting is the same as $\mathrm{SU}(N)/\mathbb{Z}_N$ case except for $D_{2N}/\mathbb{Z}_2 \times \mathbb{Z}_2$.

$D_{2N+1}/\mathbb{Z}_4$, $E_6/\mathbb{Z}_3$ and $E_7/\mathbb{Z}_2$. The counting of $D_{2N+1}/\mathbb{Z}_4$, $E_6/\mathbb{Z}_3$ and $E_7/\mathbb{Z}_2$ cases is the same as those of $\mathrm{SU}(4)/\mathbb{Z}_4$, $\mathrm{SU}(3)/\mathbb{Z}_3$ and $\mathrm{SU}(2)/\mathbb{Z}_2$ cases respectively, because our counting only depends on the center $\Gamma_{\mathcal{G}}$.

- E_8 E_8 has only one orbit and the result is

$$n_t^1 = 1\,. \tag{2.37}$$

  This result is just a reflection of the well-known fact that E_8 lattice is self-dual.

- $D_{2N}/\mathbb{Z}_2 \times \mathbb{Z}_2$ $D_{2N}/\mathbb{Z}_2 \times \mathbb{Z}_2$ must be classified by v^2 mod 2, because v^2 takes value on $\mathbb{Z}_2$. This classification produces degeneration $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ $((m,n) \to m+n)$. The correct counting including this degeneration is given by the following treatment. We regard the counting problem in this case as the one of $\mathrm{SU}(2)/\mathbb{Z}_2$ theory on a manifold with the replaced intersection form $H^{\oplus 11} \to H^{\oplus 22}$. This treatment reproduces the desired degeneration and the correct numbers. The result is

$$\begin{aligned} n_0^2 &= 2^{43} + 2^{21} - 1\,, \\ n_1^2 &= 2^{43} - 2^{21}\,, \\ n_t^1 &= 1\,. \end{aligned} \tag{2.38}$$

2.4 Partition function for $\mathrm{SU}(N)$ gauge theory on $K3$

Following [21, 11, 13], we derive the $\mathrm{SU}(N)$ partition function on $K3$ as a reconstruction of [21, 11, 13]. By using the picture that $\mathcal{N}=4$ U(1) gauge theory is given by single $M5$ brane wrapped around $K3 \times T^2$, one can understand that U(1) partition function on $K3$ is given by

$$\frac{1}{\eta(\tau)^{24}} =: G(\tau)\,. \tag{2.39}$$

Here we ignore the contribution from the U(1) fluxes.

To obtain $\mathcal{N}=4$ $\mathrm{U}(N)$ gauge theory on $K3$, we consider N coincident $M5$ brane wrapped around $K3 \times T^2$. Furthermore by fixing 't Hooft flux trivial $v=0$, we obtain $\mathcal{N}=4$ $\mathrm{SU}(N)$ gauge theory on $K3$. In this picture, Hecke transformation naturally appears as the effect of summing up all the ways of wrapping T^2 in $K3 \times T^2$ N times by T^2 in $M5$ brane. However in the remaining subsection, we apply a more mechanical method for considering the generalization to D, E gauge theory. Of course the result is the same as the result from the above picture.

First we consider the primary function

$$G_0(\tau) := \left(\frac{1}{\eta(\tau/N)}\right)^{24}, \tag{2.40}$$

as a piece of $Z_{\mathrm{SU}(N)}(\tau)$. Next, by transforming $G_0(\tau)$ by $\mathrm{SL}(2,\mathbb{Z})$, we obtain the other pieces of $Z_{\mathrm{SU}(N)}(\tau)$. Following [21], we introduce some combination of the above functions $Z_j(\tau)$ corresponding to the $\mathrm{SU}(N)/\mathbb{Z}_N$ partition function with $v_N^2 \equiv 2j$ mod $2N$. By requiring Montonen-Olive duality (2.8) between $Z_t(\tau)$ and $Z_{\mathrm{SU}(N)/\mathbb{Z}_N}$, we can determine the coefficients of G's in $Z_j(\tau)$.

Let us move to writing out the concrete SU(N) partition function for three cases, prime N, non-prime $N = p^m$ and $N = pq$.

A_{N-1}**:** N **prime.** By transforming $G_0(\tau)$ by SL$(2,\mathbb{Z})$, we obtain $N+1$ functions,

$$G_j(\tau) := \left(\frac{1}{\eta(\tau + j/N)}\right)^{24}, \qquad j = 0, \dots, N-1\,, \tag{2.41}$$

$$H_0(\tau) := \left(\frac{1}{\eta(N\tau)}\right)^{24}. \tag{2.42}$$

Following [21], we define the following partition functions,

$$Z_j(\tau) := \frac{1}{N}\Big(G_0(\tau) + \zeta_N^{-j} G_1(\tau) + \cdots + \zeta_N^{-j(N-1)} G_{N-1}(\tau)\Big), \quad j = 0, \dots, N-1\,, \tag{2.43}$$

$$Z_t(\tau) := \frac{1}{N^2} H_0(\tau) + Z_0(\tau)\,, \tag{2.44}$$

From these partition functions and the numbers in section 2.3, we obtain

$$Z_{\mathrm{SU}(N)/\mathbb{Z}_N}(\tau) := \sum_v Z_v(\tau) = n_0 Z_0(\tau) + \cdots + n_{N-1} Z_{N-1}(\tau) + n_t Z_t(\tau)\,. \tag{2.45}$$

Eqs. (2.44) and (2.45) satisfy the Montonen-Olive duality (2.6). We will call the functions appeared in the partition functions as G functions. Most of the G functions are obtained from SL$(2,\mathbb{Z})$ transformation of $G_0(\tau)$, but some G functions must be added such as in the following non-prime N cases.

Determination of the partition function for non-prime N case has the same structure as that of the counting the number of orbits. The result is the following.

A_{N-1}**:** N **non-prime.**

(i) $N = p^m$ *case.* We introduce the following G functions of block p^M and type j:

Block p^M, $(M = m, m-1, \dots, 0)$,

$$G_j^{p^M}(\tau) := \left(\frac{1}{\eta(\frac{p^{m-M}\tau + j}{p^M})}\right)^{24}, \qquad j = 0, \dots, p^M - 1\,. \tag{2.46}$$

Note that most of these functions are obtained by SL$(2,\mathbb{Z})$ transformation from $G_0^{p^m}(\tau)$. However we have to introduce some G functions (which are not obtained by SL$(2,\mathbb{Z})$ transformation from $G_0^{p^m}(\tau)$), so that we obtain the set of partition functions corresponding to all the types considered in section 2.3.

By using $G_j^{p^m}(\tau)$, $(j = 0, \dots, p^m - 1)$, we introduce the partition function of block p^m and type j:

$$Z_j^{p^m}(\tau) := \frac{1}{p^m}\left(G_0^{p^m}(\tau) + \zeta_{p^m}^{-j} G_1^{p^m}(\tau) + \cdots + \zeta_{p^m}^{-j(p^m-1)} G_{p^m-1}^{p^m}(\tau)\right), \qquad j = 0, \dots, p^m - 1\,. \tag{2.47}$$

For the partition function of block p^M, $(M = m-1, \ldots, 1)$ and type j, we introduce

$$
\begin{aligned}
Z^{p^M}_{lp^{M-1}+k}(\tau) := \frac{1}{p^{2m-M}} \Big(G^{p^M}_0(\tau) + \zeta_{p^M}^{-(lp^{M-1}+k)} G^{p^M}_1(\tau) + \cdots \\
\cdots + \zeta_{p^M}^{-(lp^{M-1}+k)(p^M-1)} G^{p^M}_{p^M-1}(\tau) \Big) + Z^{p^{M+1}}_{kp^2}(\tau) \,, \\
l = 0, \ldots, p-1 \,, \quad k = 0, \ldots, p^{M-1}-1 \,.
\end{aligned} \tag{2.48}
$$

Note that the partition function $Z^{p^M}_{lp^{M-1}+k}(\tau)$ consists of $G^{p^M}_j(\tau)$, $(M = 0, \ldots, p^M-1)$, and the partition function $Z^{p^{M+1}}_{kp^2}(\tau)$. This is the same reason why the number of orbits of block p^l is calculated by the naive number of orbits of block p^l and that of block p^{l-1}.

Finally for the partition function of block 1, we introduce

$$
\begin{aligned}
Z^1_t(\tau) &:= \frac{1}{p^{2m}} G^1_0(\tau) + Z^p_0(\tau) \\
&= \frac{1}{p^{2m}} \sum_{\substack{0 \le a,b,d \in \mathbb{Z} \\ ad = p^m, b<d}} dG\left(\frac{a\tau+b}{d}\right).
\end{aligned} \tag{2.49}
$$

In the same way as the prime N case, we obtain

$$
\begin{aligned}
Z_{\mathrm{SU}(p^m)/\mathbb{Z}_{p^m}}(\tau) &:= \sum_{v_N} Z_{v_N}(\tau) \\
&= n^{p^m}_0 Z^{p^m}_0(\tau) + \cdots + n^{p^m}_{p^m-1} Z^{p^m}_{p^m-1}(\tau) + \\
&\quad + n^{p^{m-1}}_0 Z^{p^{m-1}}_0(\tau) + \cdots + n^{p^{m-1}}_{p^{m-1}-1} Z^{p^{m-1}}_{p^{m-1}-1}(\tau) + \\
&\quad + \cdots + n^p_0 Z^p_0(\tau) + \cdots + n^p_{p-1} Z^p_{p-1}(\tau) + n^1_t Z^1_t(\tau) \\
&= \frac{1}{p^{2m}} \sum_{\substack{0 \le a,b,d \in \mathbb{Z} \\ ad = p^m, b<d}} d^{12} (\gcd(b,d))^{11} G\left(\frac{a\tau+b}{d}\right).
\end{aligned} \tag{2.50}
$$

Last equalities in (2.49) and (2.50) are nothing but the result in [11].

(ii) $N = pq$ *case.* We introduce the following G functions by $\mathrm{SL}(2,\mathbb{Z})$ transformation from $G^{pq}_0(\tau)$,

$$
G^{pq}_j(\tau) := \left(\frac{1}{\eta(\frac{\tau+j}{pq})}\right)^{24}, \qquad j = 0, \ldots, pq-1 \,. \tag{2.51}
$$

Block p,

$$
G^{p}_j(\tau) := \left(\frac{1}{\eta(\frac{q\tau+j}{p})}\right)^{24}, \qquad j = 0, \ldots, p-1 \,, \tag{2.52}
$$

Block q,

$$
G^{q}_j(\tau) := \left(\frac{1}{\eta(\frac{p\tau+j}{q})}\right)^{24}, \qquad j = 0, \ldots, q-1 \,, \tag{2.53}
$$

Block 1,

$$G_t^1(\tau) := \left(\frac{1}{\eta(pq\tau)}\right)^{24}. \qquad (2.54)$$

For the partition function of block pq, we introduce

$$Z_j^{pq}(\tau) := \frac{1}{pq}\Big(G_0^{pq}(\tau) + \zeta_{pq}^{-j}G_1^{pq}(\tau) + \cdots + \zeta_{pq}^{-j(pq-1)}G_{pq-1}^{pq}(\tau)\Big), \qquad j = 0,\ldots,pq-1\,, \qquad (2.55)$$

By imitating the counting of n_j^{pq}, we introduce the partition function of block p,

$$\begin{aligned} Z_{l_k^p}^p(\tau) \; &:= \; \frac{1}{pq^2}\left(G_0^p(\tau) + \zeta_p^{-(l_k^p)}G_1^p(\tau) + \cdots + \zeta_p^{-(l_k^p)(p-1)}G_{p-1}^p(\tau)\right) + \\ &\quad + Z_k^{pq}(\tau)\,, \qquad 0 \le l_k^p \le p-1\,, \end{aligned} \qquad (2.56)$$

where l_k^p satisfies $l_k^p q^2 \equiv 2qk \bmod 2pq$. The partition function of block q are similarly introduced.

Finally for block 1, we introduce

$$\begin{aligned} Z_t^1(\tau) \; &= \; \frac{1}{p^2q^2}G_t^1(\tau) + Z_0^p(\tau) + Z_0^q(\tau) - Z_0^{pq}(\tau) \\ &= \; \frac{1}{p^2q^2}\sum_{\substack{0\le a,b,d\in\mathbb{Z}\\ ad=pq,b<d}} dG\left(\frac{a\tau+b}{d}\right). \end{aligned} \qquad (2.57)$$

$$\begin{aligned} Z_{\mathrm{SU}(pq)/\mathbb{Z}_{pq}}(\tau) &:= \sum_{v_N} Z_{v_N}(\tau) \\ &= n_0^{pq}Z_0^{pq}(\tau) + \cdots + n_{pq-1}^{p^m}Z_{pq-1}^{p^m}(\tau) + n_0^pZ_0^p(\tau) + \cdots + n_{p-1}^pZ_{p-1}^p(\tau) + \\ &\quad + n_0^qZ_0^q(\tau) + \cdots + n_{q-1}^qZ_{q-1}^{q-1}(\tau) + n_t^1Z_t^1(\tau) \\ &= \frac{1}{p^2q^2}\sum_{\substack{0\le a,b,d\in\mathbb{Z}\\ ad=pq,b<d}} d^{12}(\gcd(b,d))^{11}G\left(\frac{a\tau+b}{d}\right). \end{aligned} \qquad (2.58)$$

Eqs. (2.57) and (2.58) are the same as the result in [11].

3. Stringy point of view

In this section, we consider the origin of the duality conjecture on the $\mathcal{N}=4$ Yang-Mills theory from the stringy point of view. Next, we estimate the general structure of the partition function in the D, E case, based on this observation. In [22], the duality conjecture on gauge theory on $\mathbb{R}^4$ is interpreted as the T-duality of type II A string theory compactified on $T^2 \times$ (ALE space). In this context, gauge group is determined by the type of the ALE space, which is classified by the ADE sub-group of SU(2), and the gauge coupling is identified with the deformation parameter τ of the complex structure of T^2. In this way, S-duality transformation of the Yang-Mills theory is derived as the T-duality transformation of T^2.

Now, we apply this interpretation to the $\mathcal{N}=4$ ADE Yang-Mills theory on $K3$ surface. We can naturally see that the corresponding string theory is the type II A string theory compactified on $K3 \times T^2 \times (ALE$ space). In this case, we have to note that both $K3$ and ALE space are hyper Kähler surfaces. Therefore, we can choose a surface of which we take zero-volume limit. If we shrink the $T^2 \times (ALE$ space), we obtain the $\mathcal{N} = 4$ ADE Yang-Mills theory on $K3$ surface. Then what happens if we shrink $T^2 \times K3$ first? We can speculate naturally that the resulting effective theory is the $\mathcal{N} = 4$ gauge theory on ALE space. Then what we have to do next is determination of the gauge group. This can be derived in the following way. Let us assume that $K3$ surface has the elliptic fibration. Generically, elliptic fibered $K3$ has 24 singular fibers. Each singular fiber has a simple nodal singularity and generate U(1) gauge symmetry. Therefore, we can see that the resulting gauge group is $\mathrm{U}(1)^{24}$. This result is quite natural if we remember the duality between type II A string theory on $K3$ and heterotic string theory on T^4 [5]. Since $\mathcal{N} = 4$ gauge theory on $K3$ does not depend on the complex structure of $K3$, we don't have to consider the gauge symmetry enhancement. In sum, we have obtained the following picture:

$$(ADE \text{ Yang-Mills theory on } K3) \simeq (\mathrm{U}(1)^{24} \text{ Yang-Mills theory on } ADE\ ALE \text{ space})\,. \tag{3.1}$$

Now, we derive the partition function of $\mathcal{N} = 4$ U(1) gauge theory on the ALE space. We can obtain this theory as a M5-brane world volume theory wrapped on $T^2 \times (ALE$ space). Appearance of U(1) gauge symmetry corresponds to the case of singly wrapped brane. According to [13, 2], the partition function of singly wrapped M5-brane is determined by the H^2 lattice and Euler number of the ALE space. Let us denote the ALE space associated with ADE group $\mathcal{G}$ by $ALE(\mathcal{G})$. It is well known that

$$H^2(ALE(\mathcal{G}), \mathbb{Z}) = -(\Lambda(\mathcal{G}))\,, \qquad \chi(ALE(\mathcal{G})) = r + 1\,, \tag{3.2}$$

where $-(\Lambda(\mathcal{G}))$ represents the root lattice whose metric is given by the minus of the standard one. Then the partition function $Z^{ALE(\mathcal{G})}_{\mathrm{U}(1)}(\tau)$ is given by

$$\begin{aligned} Z^{ALE(\mathcal{G})}_{\mathrm{U}(1)}(\tau) &= \frac{\theta^0_{\mathcal{G}}(\tau)}{\eta(\tau)^{r+1}}\,, \\ \theta^0_{\mathcal{G}}(\tau) &:= \sum_{m \in \mathbb{Z}^r} q^{\frac{1}{2}{}^t(m) C_{\mathcal{G}}(m)}\,. \end{aligned} \tag{3.3}$$

where $q := \exp(2\pi i \tau)$, and $C_{\mathcal{G}}$ is Cartan Matrix of $\mathcal{G}$. Notice that the above result is compatible with the result of Nakajima [19], that asserts that direct sum of the cohomology of the moduli space of U(1) instantons on $ALE(\mathcal{G})$ is decomposed into level 1 representations of the affine Lie algebra of $\mathcal{G}$. Combining (3.1) and (3.3), we estimate (at least as the first approximation) that the partition function $Z^{K3}_{\mathcal{G}}(\tau)$ of $\mathcal{N} = 4$ $\mathcal{G}$ Yang-Mills theory on $K3$ surface has the following form:

$$Z^{K3}_{\mathcal{G}}(\tau) = \left(\frac{\theta^0_{\mathcal{G}}(\tau)}{\eta(\tau)^{r+1}} \right)^{24}\,. \tag{3.4}$$

4. Denominator identity and partition function for D, E gauge groups on $K3$

In this section, we turn into another suggestive result useful for determination of the partition function of D, E gauge theory. In our previous work [7], we came across a mysterious identity:

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta^2_{A_{N-1}}(\tau)}{\eta(\tau)^N}, \tag{4.1}$$

where $\theta^2_{A_{N-1}}(\tau)$ is a kind of A_{N-1} theta function which is explicitly defined for general ADE groups as follows:

Definition. *For rank r gauge group $\mathcal{G}$, which has dual Coxeter number h, half of the sum of the positive roots ρ and roots $\{\alpha_k\}$ (see [8]), we define the following function*

$$\theta^2_{\mathcal{G}}(\tau) := \sum_{m \in \mathbb{Z}^r} q^{\frac{1}{2}{}^t(m+\frac{\rho}{h})C_{\mathcal{G}}(m+\frac{\rho}{h})}, \tag{4.2}$$

where $q := \exp(2\pi i\tau)$.

Since the identity (4.1) originates from the celebrated denominator identity of the affine Lie algebra, it is straightforward to generalize (4.1) to D, E case:

Theorem. *(Macdonald, Kac) [12, 8]*

$$\sum_{m \in \mathbb{Z}^r} q^{\frac{1}{2}{}^t(m+\frac{\rho}{h})C_{\mathcal{G}}(m+\frac{\rho}{h})} = q^{\frac{1}{24}\frac{\dim(\mathcal{G})}{h}} \prod_{n=1}^{\infty} \left(1 - q^{\frac{n}{h}}\right)^r \prod_{\alpha_k} \left(1 - q^{\frac{n}{h}} \zeta_h^{\langle \rho, \alpha_k \rangle}\right), \tag{4.3}$$

where $\zeta_h := \exp(2\pi i/h)$.

Furthermore as will be shown in appendix B, the r.h.s. of (4.3) can be expressed as η-product.

With the consideration in the previous section, we had better regard this identity as the one that interpolate the previous estimation of the partition function in (3.4) to the well-known form given by η-product. Of course, we have to note that the theta function appearing in (4.3) is not $\theta^0_{\mathcal{G}}(\tau)$ but $\theta^2_{\mathcal{G}}(\tau)$. This subtle translation by ρ/h in the theta sum is puzzling, and seems to be caused by boundary condition of the ALE space. This point should be pursued further, but instead, we consider here how to use the identity (4.3). To this aim, we look back at the construction of A_N partition function given in section 2. In section 2, we used $G_0(\tau) = (\theta^2_{A_{N-1}}(\tau)/\eta(\tau)^N)^{24}$ as the primary building block and summed up the $\mathrm{SL}(2,\mathbb{Z})$ orbit of $(\theta^2_{A_{N-1}}(\tau)/\eta(\tau)^N)^{24}$ so that the whole partition functions satisfy the duality conjecture. Therefore, we expect that the identity (4.3) tells us how to modify the first approximation (3.4) of the D, E partition function on $K3$. The reason why the A_N partition function is written in the form of η-product comes from the fact that the moduli space of $\mathrm{SU}(N)$ instanton on $K3$ is described by Hilbert scheme of points on $K3$. This fact can be derived by Fourier-Mukai transform of semi-stable sheaves on $K3$. Since we treat D, E type instanton on the same $K3$, we speculate that some analogous structure may exist in the moduli space of D, E type instanton on $K3$. In other words, the primary building block of the D, E partition function should take the form of η-product. Here, we show the table of the η-products obtained from the identity (4.3).

With these consideration, we choose for the primary function of D, E gauge groups,

$$G_0(\tau) = \left(\frac{\theta_{\mathcal{G}}^2(\tau)}{\eta(\tau)^{r+1}}\right)^{24}. \tag{4.4}$$

$\mathcal{G}$	$\theta_{\mathcal{G}}^2(\tau)/\eta(\tau)^{r+1}$
A_r	$\dfrac{1}{\eta(\frac{\tau}{r+1})}$
D_r	$\dfrac{\eta(\frac{\tau}{r-1})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{2r-2})}$
E_6	$\dfrac{\eta(\frac{\tau}{4})\eta(\frac{\tau}{6})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{12})}$
E_7	$\dfrac{\eta(\frac{\tau}{6})\eta(\frac{\tau}{9})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{18})}$
E_8	$\dfrac{\eta(\frac{\tau}{6})\eta(\frac{\tau}{10})\eta(\frac{\tau}{15})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{5})\eta(\frac{\tau}{30})}$

Table 2: η-products of $\mathcal{G}$.

Because these functions have the form of η-product, we can repeat the same process in section 2.4 for these primary functions. As a result, we obtain the partition function for D, E gauge groups on $K3$, which satisfy Montonen-Olive duality (2.8). Though our computation stems from a wild speculation, it is quite non-trivial that the resulting partition functions satisfy Montonen-Olive duality condition in all the cases we test.

Before moving to the individual case, we will sketch the processes of the determination of D, E partition function on $K3$. First we have to prepare the sets of G functions. Here we point out that each D, E theory has the corresponding $\mathrm{SU}(N)$ theory characterized by its center. We prepare G functions so that all the G functions reproduce the corresponding $\mathrm{SU}(N)$ structure. Most of G's are obtained by $\mathrm{SL}(2, \mathbb{Z})$ transformation of $G_0(\tau)$, but some G functions are added in order to satisfy the above condition. Basically one can obtain D, E partition function on $K3$ by imposing Montonen-Olive duality on some combination of G functions. However we also introduce the intermediate functions $\tilde{G}$, which are some combination of G functions. Here $\mathrm{SL}(2, \mathbb{Z})$ transformation of $\tilde{G}$ functions is the same as that of G functions of the corresponding $\mathrm{SU}(N)$ theory.

4.1 D_{2N}

D_{2N} is dual to $D_{2N}/\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus we introduce 't Hooft flux $v \in H^2(K3, \mathbb{Z}_2 \times \mathbb{Z}_2)$. As we have already saw in section 2.3, $Z_v(\tau)$ are classified by degenerated $\mathbb{Z}_2$. In this subsection, we try to derive D_2 and D_4 explicitly.

D_2**.** Following the SU(2) case, we first introduce the G functions obtained from transforming $G_0(\tau)$ by $\mathrm{SL}(2, \mathbb{Z})$.

$$G_j(\tau) := \left(\frac{\eta(\tau)}{\eta(\frac{\tau+j}{2})^2}\right)^{24}, \qquad j = 0, 1, \tag{4.5}$$

$$H_0(\tau) := \left(\frac{\eta(\tau)}{\eta(2\tau)^2}\right)^{24}. \tag{4.6}$$

To define the partition function for even type ($v^2 \equiv 0$) and odd type($v^2 \equiv 2$), we introduce

$$Z_j(\tau) := \frac{1}{4}\left(G_0(\tau) + (-1)^j G_1(\tau)\right), \qquad j = 0, 1. \tag{4.7}$$

We also introduce the partition function of trivial type,

$$Z_t(\tau) := \frac{1}{16}H_0(\tau) + Z_0(\tau)\,. \tag{4.8}$$

By using (4.7) and (4.8) and the numbers $\{n_j\}$ in section 2.3, we obtain

$$Z_{D_2/\mathbb{Z}_2\times\mathbb{Z}_2}(\tau) := n_0 Z_0(\tau) + n_1 Z_1(\tau) + n_t Z_t(\tau)\,. \tag{4.9}$$

Eqs. (4.8) and (4.9) surely satisfy Montonen-Olive duality (2.8),

$$Z_t\left(-\frac{1}{\tau}\right) = 4^{-11}\left(\frac{\tau}{i}\right)^{-12} Z_{D_2/\mathbb{Z}_2\times\mathbb{Z}_2}(\tau)\,. \tag{4.10}$$

Note that the coefficients of G's in (4.7) and (4.8) are determined by imposing (4.10).

D_4**.** Following the D_2 case, we first introduce the G functions obtained from transforming $G_0(\tau)$ by $\mathrm{SL}(2,\mathbb{Z})$.

$$\begin{aligned}
G_j(\tau) &:= \left(\frac{\eta(\frac{\tau+j}{3})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{6})}\right)^{24}, \qquad j = 0,\ldots,5\,, && (4.11)\\
H_j(\tau) &:= \left(\frac{\eta(\frac{\tau+j}{3})}{\eta(2\tau)\eta(\frac{2\tau+2j}{3})}\right)^{24}, \qquad j = 0,\ldots,2\,, && (4.12)\\
I_j(\tau) &:= \left(\frac{\eta(3\tau)}{\eta(\frac{\tau+j}{2})\eta(\frac{3\tau+3j}{2})}\right)^{24}, \qquad j = 0,1\,, && (4.13)\\
J_0(\tau) &:= \left(\frac{\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^{24}. && (4.14)
\end{aligned}$$

Contrary to the D_2 case, we will introduce the intermediate $\tilde{G}(\tau)$ functions, so that we pick up the powers $q^{(1/2)\mathbb{Z}}$ from the G functions having $q^{(1/6)\mathbb{Z}}$ powers,

$$\begin{aligned}
\tilde{G}_j(\tau) &:= G_j(\tau) + G_{2+j}(\tau) + G_{4+j}(\tau) + I_j(\tau)\,, \qquad j = 0,1\,, && (4.15)\\
\tilde{H}_0(\tau) &:= H_0(\tau) + H_1(\tau) + H_2(\tau) + J_0(\tau)\,. && (4.16)
\end{aligned}$$

Once we introduce $\tilde{G}(\tau)$'s, we only follow the same processes as D_2 case. Actually the modular property of $\tilde{G}_j(\tau)$ and $\tilde{H}_0(\tau)$ is completely the same as that of $G_j(\tau)$ and $H_0(\tau)$ in D_2.

$$\begin{aligned}
Z_j(\tau) &:= \frac{1}{4}\left(\tilde{G}_0(\tau) + (-1)^j\tilde{G}_1(\tau)\right), \qquad j = 0,1\,, && (4.17)\\
Z_t(\tau) &:= \frac{1}{16}\tilde{H}_0(\tau) + Z_0(\tau)\,, && (4.18)\\
Z_{D_4/\mathbb{Z}_2\times\mathbb{Z}_2}(\tau) &:= n_0 Z_0(\tau) + n_1 Z_1(\tau) + n_t Z_t(\tau)\,, && (4.19)
\end{aligned}$$

Eqs. (4.18) and (4.19) also satisfy Montonen-Olive duality (2.8),

$$Z_t\left(-\frac{1}{\tau}\right) = 4^{-11}\left(\frac{\tau}{i}\right)^{-12} Z_{D_4/\mathbb{Z}_2\times\mathbb{Z}_2}(\tau)\,. \tag{4.20}$$

4.2 D_{2N+1}

D_{2N+1} is dual to $D_{2N+1}/\mathbb{Z}_4$. Thus we introduce 't Hooft flux $v \in H^2(K3, \mathbb{Z}_4)$. As we have already saw in section 2.3, $Z_v(\tau)$ are classified by $\mathbb{Z}_4$. This structure is the same as A_3 case. Indeed

$$D_3 = A_3 \,.$$

In this subsection, we try to derive the D_5 partition function explicitly.

D_5**.** Following the SU(4) case, we first introduce the G functions obtained from transforming $G_0(\tau)$ by $\mathrm{SL}(2,\mathbb{Z})$, and add some G functions so that all G functions form the modular property of SU(4).

$$G_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{4})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{8})}\right)^{24}, \qquad j = 0,\ldots,7\,, \tag{4.21}$$

$$H_j(\tau) := \left(\frac{\eta(\frac{2\tau+j}{2})}{\eta(2\tau)\eta(\frac{2\tau+j}{4})}\right)^{24}, \qquad j = 0,\ldots,3\,, \tag{4.22}$$

$$I_0(\tau) := \left(\frac{\eta(4\tau)}{\eta(2\tau)\eta(\frac{4\tau+1}{2})}\right)^{24}, \tag{4.23}$$

$$J_0(\tau) := \left(\frac{\eta(4\tau)}{\eta(2\tau)\eta(8\tau)}\right)^{24}, \tag{4.24}$$

$$K_j(\tau) := \left(\frac{\eta(2\tau)}{\eta(\frac{2\tau+j}{2})^2}\right)^{24}, \qquad j = 0,1\,, \tag{4.25}$$

$$L_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{2})}{\eta(\frac{\tau+j}{4})^2}\right)^{24}, \qquad j = 0,\ldots,3\,, \tag{4.26}$$

$$M_0(\tau) := \left(\frac{\eta(2\tau)}{\eta(4\tau)^2}\right)^{24}. \tag{4.27}$$

Precisely speaking, G_j, $(j = 0,\ldots,7)$, H_1, H_3, I_0, J_0 are obtained from transforming $G_0(\tau)$ by $\mathrm{SL}(2,\mathbb{Z})$, and H_0, H_2, K_j, $(j = 0,1)$, L_j, $(j = 0,\ldots,3)$, M_0 are obtained from requiring the modular property of SU(4).

For the same reason as the D_4 case, we will introduce the intermediate $\tilde{G}(\tau)$ functions, so that we pick up the powers $q^{(1/4)\mathbb{Z}}$ from G functions,

$$\tilde{G}^4_j(\tau) := G_j(\tau) + G_{4+j}(\tau) + 2^{12}L_j(\tau)\,, \qquad j = 0,\ldots,3\,, \tag{4.28}$$

$$\tilde{G}^2_j(\tau) := H_j(\tau) + H_{2+j}(\tau) + 2^{12}K_j\,, \qquad j = 0,1\,, \tag{4.29}$$

$$\tilde{G}^1_t(\tau) := 2^{12}I_0(\tau) + J_0(\tau) + M_0(\tau)\,. \tag{4.30}$$

The modular property of $\tilde{G}^4_j(\tau)$, $\tilde{G}^2_j(\tau)$ and $\tilde{G}^1_t(\tau)$ is completely the same as that of $G^4_j(\tau)$, $G^2_j(\tau)$and $G^1_t(\tau)$ in SU(4) respectively. Mysterious 2^{12} factor in (4.28), (4.29) and (4.30) is needed for this equivalence. By using $\tilde{G}(\tau)$'s, we follow the same processes as the SU(4) case.

$$Z_j^4(\tau) := \frac{1}{4}\left(\tilde{G}_0^4(\tau) + \zeta_4^{-j}\tilde{G}_1^4(\tau) + \cdots + \zeta_4^{-j3}\tilde{G}_3^1(\tau)\right), \qquad j = 0, \ldots, 3, \tag{4.31}$$

$$Z_j^2(\tau) := \frac{1}{8}\left(\tilde{G}_0^2(\tau) + (-1)^j\tilde{G}_1^2(\tau)\right) + Z_0^4(\tau), \qquad j = 0, 1, \tag{4.32}$$

$$Z_t^1(\tau) := \frac{1}{16}\tilde{G}_t^1(\tau) + Z_0^2(\tau), \tag{4.33}$$

$$Z_{D_5/\mathbb{Z}_4}(\tau) := n_0^4 Z_0^4(\tau) + \cdots + n_3^4 Z_{N-1}^4(\tau) + n_0^2 Z_0^2(\tau) + n_1^2 Z_1^2(\tau) + n_t^1 Z_t^1(\tau). \tag{4.34}$$

Eq. (4.33) and (4.34) indeed satisfy Montonen-Olive duality (2.8) in the same way as SU(4) case,

$$Z_t^1\left(-\frac{1}{\tau}\right) = 4^{-11}\left(\frac{\tau}{i}\right)^{-12} Z_{D_5/\mathbb{Z}_4}(\tau). \tag{4.35}$$

4.3 $E_{6,7,8}$

E_6. E_6 is dual to $E_6/\mathbb{Z}_3$. Thus we introduce 't Hooft flux $v \in H^2(K3, \mathbb{Z}_3)$. As we already saw in section 2.3, $Z_v(\tau)$ are classified by $\mathbb{Z}_3$. Explicit G functions are written in appendix C. Following the D_4 and D_5 case, we introduce the intermediate functions,

$$\tilde{G}_j(\tau) := \sum_{k=0}^{3} G_{3k+j}(\tau) + \sum_{k=0}^{1} H_{3k+3-j}(\tau) + \sum_{k=0}^{1} N_{3k+j}(\tau) + J_{3-j}(\tau), \qquad j = 0, 1, 2, \tag{4.36}$$

$$\tilde{H}_0(\tau) := \sum_{k=0}^{3} I_k(\tau) + \sum_{k=0}^{1} K_k(\tau) + \sum_{k=0}^{1} M_k(\tau) + L_0(\tau). \tag{4.37}$$

By using $\tilde{G}(\tau)$'s, we follow the same process as the SU(3) case.

$$Z_j(\tau) := \frac{1}{3}\left(\tilde{G}_0(\tau) + \zeta_3^{-j}\tilde{G}_1(\tau) + \zeta_3^{-j2}\tilde{G}_2(\tau)\right), \qquad j = 0, 1, 2, \tag{4.38}$$

$$Z_t(\tau) := \frac{1}{9}\tilde{H}_0(\tau) + Z_0(\tau), \tag{4.39}$$

$$Z_{E_6/\mathbb{Z}_3}(\tau) := n_0 Z_0(\tau) + n_1 Z_1(\tau) + n_2 Z_2(\tau) + n_t Z_t(\tau). \tag{4.40}$$

Eqs. (4.39) and (4.40) surely satisfy Montonen-Olive duality (2.8) in the same way as the SU(3) case,

$$Z_t\left(-\frac{1}{\tau}\right) = 3^{-11}\left(\frac{\tau}{i}\right)^{-12} Z_{E_6/\mathbb{Z}_3}(\tau). \tag{4.41}$$

E_7. By repeating the similar processes to E_6 case, the results of E_7 are given by

$$\tilde{G}_j(\tau) := \sum_{k=0}^{8} G_{2k+j}(\tau) + \sum_{k=0}^{2} I_{2k+j}(\tau) + \sum_{k=0}^{2} N_{2k+j}(\tau) + K_j(\tau), \qquad j = 0, 1, \tag{4.42}$$

$$\tilde{H}_0(\tau) := \sum_{k=0}^{8} H_k(\tau) + \sum_{k=0}^{3} J_k(\tau) + \sum_{k=0}^{2} M_k(\tau) + L_0(\tau). \tag{4.43}$$

Corresponding SU(N) theory is SU(2). Thus we define

$$Z_j(\tau) := \frac{1}{2}\left(\tilde{G}_0(\tau) + (-1)^{-j}\tilde{G}_1(\tau)\right), \qquad j = 0, 1, \tag{4.44}$$

$$Z_t(\tau) := \frac{1}{4}\tilde{H}_0(\tau) + Z_0(\tau), \tag{4.45}$$

$$Z_{E_7/\mathbb{Z}_2}(\tau) := n_0 Z_0(\tau) + n_1 Z_1(\tau) + n_t Z_t(\tau). \tag{4.46}$$

Eqs. (4.45) and (4.46) satisfy Montonen-Olive duality (2.8) in the same way as the SU(2) case,

$$Z_t\left(-\frac{1}{\tau}\right) = 2^{-11}\left(\frac{\tau}{i}\right)^{-12} Z_{E_7/\mathbb{Z}_2}(\tau). \tag{4.47}$$

E_8. Since E_8 is self-dual group, $Z_t(\tau)$ turns out to be a modular form of weight -12. In this case, we introduce G functions obtained from transforming G_0 by SL$(2,\mathbb{Z})$.

$$G_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{6})\eta(\frac{\tau+j}{10})\eta(\frac{\tau+j}{15})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})\eta(\frac{\tau+j}{5})\eta(\frac{\tau+j}{30})}\right)^{24}, \qquad j = 0, \ldots, 29, \tag{4.48}$$

$$H_j(\tau) := \left(\frac{\eta(\frac{2\tau+2j}{3})\eta(\frac{2\tau+2j}{5})\eta(\frac{\tau+j}{15})}{\eta(2\tau)\eta(\frac{\tau+j}{3})\eta(\frac{\tau+j}{5})\eta(\frac{2\tau+2j}{15})}\right)^{24}, \qquad j = 0, \ldots, 14, \tag{4.49}$$

$$I_j(\tau) := \left(\frac{\eta(\frac{3\tau+3j}{2})\eta(\frac{\tau+j}{10})\eta(\frac{3\tau+3j}{5})}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(\frac{\tau+j}{5})\eta(\frac{3\tau+3j}{10})}\right)^{24}, \qquad j = 0, \ldots, 9, \tag{4.50}$$

$$J_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{6})\eta(\frac{5\tau+5j}{2})\eta(\frac{5\tau+5j}{3})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})\eta(5\tau)\eta(\frac{5\tau+5j}{6})}\right)^{24}, \qquad j = 0, \ldots, 5, \tag{4.51}$$

$$K_j(\tau) := \left(\frac{\eta(6\tau)\eta(\frac{2\tau+2j}{5})\eta(\frac{3\tau+3j}{5})}{\eta(2\tau)\eta(3\tau)\eta(\frac{\tau+j}{5})\eta(\frac{6\tau+6j}{5})}\right)^{24}, \qquad j = 0, \ldots, 4, \tag{4.52}$$

$$L_j(\tau) := \left(\frac{\eta(\frac{2\tau+2j}{3})\eta(10\tau)\eta(\frac{5\tau+5j}{3})}{\eta(2\tau)\eta(\frac{\tau+j}{3})\eta(5\tau)\eta(\frac{10\tau+10j}{3})}\right)^{24}, \qquad j = 0, 1, 2, \tag{4.53}$$

$$M_j(\tau) := \left(\frac{\eta(\frac{3\tau+j}{2})\eta(\frac{5\tau+j}{2})\eta(15\tau)}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(5\tau)\eta(\frac{15\tau+j}{2})}\right)^{24}, \qquad j = 0, 1, \tag{4.54}$$

$$N_0(\tau) := \left(\frac{\eta(6\tau)\eta(10\tau)\eta(15\tau)}{\eta(2\tau)\eta(3\tau)\eta(5\tau)\eta(30\tau)}\right)^{24}, \tag{4.55}$$

Each G function is transformed to each other by SL$(2,\mathbb{Z})$ with no extra factor except for $(\frac{\tau}{i})^{-12}$. Therefore, there is no need to introduce the intermediate functions. The partition function is simply given by adding all the G functions.

$$\begin{aligned} Z_t(\tau) := &\sum_{j=0}^{29} G_j(\tau) + \sum_{j=0}^{14} H_j(\tau) + \sum_{j=0}^{9} I_j(\tau) + \sum_{j=0}^{5} J_j(\tau) + \sum_{j=0}^{4} K_j(\tau) + \\ &+ \sum_{j=0}^{2} L_j(\tau) + \sum_{j=0}^{1} M_j(\tau) + N_0(\tau). \end{aligned} \tag{4.56}$$

Therefore, (4.56) is surely self-dual or a modular form,

$$Z_t\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{-12} Z_t(\tau). \tag{4.57}$$

Indeed $Z_t(\tau)$ can be expressed by the well-known modular forms as follows:

$$\begin{aligned} Z_t(\tau) \;=\; \frac{1}{\eta(\tau)^{24}}\Big(& j^8(\tau) - 5976 j^7(\tau) + 14049204 j^6(\tau) - 16450492296 j^5(\tau) + \\ & + 10006744823442 j^4(\tau) - 2995100782701144 j^3(\tau) + \\ & + 373127947258066100 j^2(\tau) - 12808385327808647208 j(\tau) + \\ & + 26763599994092029512\Big), \end{aligned} \tag{4.58}$$

where $j(\tau)$ is the famous modular j-function [14]. Notice that

$$j(\tau) = \chi_{1,0}(\tau)^3 \tag{4.59}$$

where $\chi_{1,0}(\tau)$ is level one E_8 character. In this way, $Z_t(\tau)$ can be expressed only in terms of $\chi_{1,0}(\tau)$ and $\eta(\tau)$. This fact ensures that $Z_t(\tau)$ is a good candidate for the E_8 partition function on $K3$.

5. Holomorphic anomaly?

In this section, we discuss that our partition function for D, E gauge group is not sufficient from the point of view of the gap condition, and search for the improvement analogous to the holomorphic anomaly on $\frac{1}{2}K3$ [13].

First, we derive the gap condition for ADE gauge groups by using the famous Atiyah-Hitchin-Singer dimension formula for the moduli space of irreducible anti-self-dual connections with ADE gauge group $\mathcal{G}$ on X [1].

$$\dim \mathcal{M}_k^{\mathcal{G}} = 4h(\mathcal{G})k - \dim \mathcal{G}\frac{\chi(X)+\sigma(X)}{2}\,, \tag{5.1}$$

where $h(\mathcal{G})$ is the dual Coxeter number, $\dim \mathcal{G}$ is the dimension of $\mathcal{G}$ and k is the instanton number. $\chi(X)$ and $\sigma(X)$ are Euler number and signature of X respectively. In our case of $X = K3$, this formula reduces to

$$\dim \mathcal{M}_k^{\mathcal{G}} = 4h(\mathcal{G})k - 4\dim \mathcal{G}\,. \tag{5.2}$$

Here we point out that $h(\mathcal{G}), \dim(\mathcal{G})$ and r are related by the following formula:

$$\dim(\mathcal{G}) = rh(\mathcal{G}) + r, h(\mathcal{G}) > r\,. \tag{5.3}$$

Therefore,

$$\dim \mathcal{M}_k^{\mathcal{G}} = 4h(\mathcal{G})(k-r) - 4r\,. \tag{5.4}$$

$\dim \mathcal{M}_k^{\mathcal{G}} \geq 0$ means $k \geq r+1$. That is, the moduli space of irreducible ASD connections with ADE gauge group on $K3$ can exist only for $k \geq r+1$. This condition restricts the form of the partition function. Remember that the partition function has the form:

$$Z_v^X(\tau) := q^{-\frac{(r+1)\chi(X)}{24}} \sum_k \chi(\mathcal{M}(v,k)) q^k \,. \tag{5.5}$$

In our case, we consider the case of $X = K3$ and of trivial flux, which corresponds to $\mathcal{M}(0,k) = \mathcal{M}_k^{\mathcal{G}}$. Thus we obtain

$$Z_t(\tau) = q^{-(r+1)} \sum_k \chi(\mathcal{M}_k^{\mathcal{G}}) q^k \,. \tag{5.6}$$

This form and the condition $k \geq r+1$ requires (5.6) to start from $\mathcal{O}(q^0)$ term. Furthermore, it is well-known that the trivial connection always exists [21]. This means that the contribution from $k=0$ appears in (5.6) for any gauge group. Therefore, forbidden q powers in (5.6) are $q^{-r}, \ldots, q^{-1}$, that is $1 \leq k \leq r$. This is the gap condition.

The $\mathrm{SU}(N)$ partition function on $K3$ surely satisfies this gap condition [21]. However our D, E partition function on $K3$ does not satisfy this gap condition. To discuss this problem, we will pick up the E_8 partition function on $K3$. The other D, E partition functions on $K3$ have the same problem as the E_8 case. First we expand $Z_t(\tau)$ of E_8:

$$\begin{aligned} Z_t(\tau) = {} & q^{-9} + 24q^{-7} + 24q^{-6} + 324q^{-5} + 600q^{-4} + 3500q^{-3} + 10008q^{-2} + \\ & + 22890627q^{-1} + 134931884553792 + 169735214436644944 0512q + \\ & + 1772938577239450377975 91920q^2 + 24442598918322165598528118 15364q^3 + \\ & + 10159099239429369126457917291280224q^4 + \cdots \,. \end{aligned} \tag{5.7}$$

What is the origin of $q^{-7}, q^{-6}, \ldots, q^{-1}$? We can suggest the following two possibilities. The one possibility is the failure of the vanishing theorem [21]. The vanishing theorem ensures the equality (5.6). From this view point, we may have derived the D, E partition function on $K3$, which corresponds to the generating function of the Euler number of the generalized moduli space including the field $A_\mu, B_{\mu\nu}^+$ and C [21, 10]. The second possibility is the appearance of the reducible connection. Atiyah-Hitchin-Singer dimension formula is the formula for irreducible ASD connection. Thus if there are reducible connections, it is no wonder why the negative powers appear in the partition function. Actually the above two possibilities are mutually related. We will discuss the second possibility more in detail here.

First we point out the well-known fact that there is a holomorphic anomaly in the $\mathrm{SU}(N)$ partition function on the manifold $b_2^+ \leq 1$. This holomorphic anomaly comes from reducible connections. We think that our situation is very similar to the holomorphic anomaly appearing in the $\mathrm{U}(N)$ partition function on $\frac{1}{2}K3$ [13, 3]. For example, let us expand the holomorphic part of the $\mathrm{U}(2)$ partition function $Z_2^{\mathrm{hol.}}(\tau)$, which does not include $E_2(\tau)$ terms [13, 3].

$$Z_2^{\mathrm{hol.}}(\tau) = \frac{1}{12} q^{-1} - 20 - 11787q - \frac{2132320}{3} q^2 + \cdots \,. \tag{5.8}$$

If one adds $E_2(\tau)$ terms, which are determined by the holomorphic anomaly equation [13], one can obtain $Z_2(\tau)$, which includes no $\mathcal{O}(q^0)$ term. Looking back at our problem, we may obtain the complete partition function of D, E gauge theory by adding $E_2(\tau)$ dependent terms, which come from reducible connections. Since $E_2(\tau)$ behave anomalously under $\mathrm{SL}(2,\mathbb{Z})$ transformation, each coefficient function should be a modular form of appropriate weight. More explicitly, expected full partition function $Z_t^{\mathrm{full}}(\tau)$ should take the following form:

$$
\begin{aligned}
Z_t^{\mathrm{full}}(\tau) &= \frac{1}{\eta(\tau)^{216}} \cdot \left(\sum_j (E_2(\tau))^j \cdot f_j(E_4, E_6) \right), \\
Z_t(\tau) &= \frac{f_0(E_4, E_6)}{\eta(\tau)^{216}}
\end{aligned}
\tag{5.9}
$$

where $f_j(E_4, E_6)$ is a modular form of weight $96 - 2j$. Of course, the above ansatz has enough degrees of freedom to cancel the coefficient of $q^{-7}, q^{-6}, \ldots, q^{-1}$ of $Z_t(\tau)$. But we don't have the principle to select the "right" $f_j(E_4, E_6)$ up to now. We expect to find the analogue of holomorphic anomaly equation used in [13]. Unfortunately, corresponding Seiberg-Witten curve is not known and the problem still needs further consideration.

6. Conclusion and discussion

We have determined the holomorphic part of the D, E partition function on $K3$, by using the D, E blow-up formula. This partition function completely satisfies Montonen-Olive duality.

Remaining problem is determination of the complete partition function of D, E gauge theory on $K3$, which satisfies the gap condition. We think that the desired complete D, E partition function on $K3$ includes the holomorphic part and the anti-holomorphic part having $E_2(\tau)$ terms. We also expect that the anti-holomorphic part of the partition function of D, E gauge theory on $K3$ should be determined by some equations, which corresponds to the holomorphic anomaly equation on $\frac{1}{2}K3$. To this aim, an approach from Seiberg-Witten curve seems to be effective since the holomorphic anomaly equation was verified from the analysis Seiberg-Witten of $\mathcal{N} = 4$ gauge theory.

Acknowledgments

We would like to thank T.Eguchi, K.Fukaya, H.Nakajima, K.Ono and K.Yoshioka for helpful suggestions and useful discussions. We owe the major part of section 5 to the discussions with them.

A. Modular property of $\mathrm{SU}(3)/\mathbb{Z}_3$ theory

In this article, we only considered the modular property of (2.8).

$$
Z_t\left(-\frac{1}{\tau}\right) = 3^{-11}\tau^{-12}\left(n_0 Z_0(\tau) + n_1 Z_1(\tau) + n_2 Z_2(\tau) + n_t Z_t(\tau)\right).
$$

Of course other modular properties in (2.6) must be considered. In this section, we derive other modular properties in (2.6) by counting the orbits for A_2 case.

A.1 $Z_0(-1/\tau)$

$$Z_0\left(-\frac{1}{\tau}\right) = 3^{-11}\tau^{-12}(m_0 Z_0(\tau) + m_1 Z_1(\tau) + m_2 Z_2(\tau) + m_t Z_t(\tau)). \tag{A.1}$$

We determine $\{m_j\}$ by using (2.6).

$$\begin{cases} m_0 = m_0^0 + \zeta_3 m_0^1 + \zeta_3^2 m_0^2 \\ m_1 = m_1^0 + \zeta_3 m_1^1 + \zeta_3^2 m_1^2 \\ m_2 = m_2^0 + \zeta_3 m_2^1 + \zeta_3^2 m_2^2 \\ m_t = m_t^0 \,, \end{cases} \tag{A.2}$$

where

$$\begin{aligned} m_j^k &= \text{number of } u^2 \equiv 2j \bmod 6 \quad \text{and} \quad V_1 \cdot u \equiv k \bmod 3, u \neq 0\,, \\ m_t^0 &= 1\,. \end{aligned} \tag{A.3}$$

Here $V_1^2 = 0$ and $V_1 \neq 0$,and $v = V_1$ corresponds to $v^2 \equiv 0 \bmod 6$ and $v \neq 0$. The results are given by

$$\begin{cases} m_0^0 = 3^{20} + 2 \cdot 3^{10} - 1 \\ m_0^1 = 3^{20} \\ m_0^2 = 3^{20} \\ m_0 = 2 \cdot 3^{10} - 1\,, \end{cases} \tag{A.4}$$

$$\begin{cases} m_1^0 = 3^{20} - 3^{10} \\ m_1^1 = 3^{20} \\ m_1^2 = 3^{20} \\ m_1 = -3^{10}\,, \end{cases} \tag{A.5}$$

$$\begin{cases} m_2^0 = 3^{20} - 3^{10} \\ m_2^1 = 3^{20} \\ m_2^2 = 3^{20} \\ m_2 = -3^{10}\,. \end{cases} \tag{A.6}$$

A.2 $Z_1(-1/\tau)$

$$Z_1\left(-\frac{1}{\tau}\right) = 3^{-11}\tau^{-12}(m_0 Z_0(\tau) + m_1 Z_1(\tau) + m_2 Z_2(\tau) + m_t Z_t(\tau))\,. \tag{A.7}$$

We determine $\{m_j\}$ by using (2.6).

$$\begin{cases} m_0 = m_0^0 + \zeta_3 m_0^1 + \zeta_3^2 m_0^2 \\ m_1 = m_1^0 + \zeta_3 m_1^1 + \zeta_3^2 m_1^2 \\ m_2 = m_2^0 + \zeta_3 m_2^1 + \zeta_3^2 m_2^2 \\ m_t = m_t^0 \,, \end{cases} \tag{A.8}$$

where

$$\begin{aligned} m_j^k &= \text{number of } u^2 \equiv 2j \bmod 6 \quad \text{and} \quad (V_1+V_2)\cdot u \equiv k \bmod 3, u \neq 0\,, \\ m_t^0 &= 1\,. \end{aligned} \tag{A.9}$$

Here $(V_1+V_2)^2 = 2$,and $v = V_1+V_2$ corresponds to $v^2 \equiv 2 \bmod 6$. The results are given by

$$\begin{cases} m_0^0 = 3^{20} - 1 \\ m_0^1 = 3^{20} + 3^{10} \\ m_0^2 = 3^{20} + 3^{10} \\ m_0 = -3^{10} - 1, \end{cases} \tag{A.10}$$

$$\begin{cases} m_1^0 = 3^{20} - 3^{10} \\ m_1^1 = 3^{20} \\ m_1^2 = 3^{20} \\ m_1 = -3^{10}, \end{cases} \tag{A.11}$$

$$\begin{cases} m_2^0 = 3^{20} + 3^{10} \\ m_2^1 = 3^{20} - 3^{10} \\ m_2^2 = 3^{20} - 3^{10} \\ m_2 = 2\cdot 3^{10}. \end{cases} \tag{A.12}$$

B. ADE blow-up formula

B.1 A_r case

It is convenient to write roots α_k of A_r algebra by orthonormal basis $\{e_j\}$. We write ρ by these basis. By using these data, we derive $\theta^2_{A_r}(\tau)$ as below.

$$\text{Roots}\{\alpha_k\}: \ e_i - e_j\,, \quad (1 \le i \neq j \le r+1)\,. \tag{B.1}$$

$$h = r+1\,, \tag{B.2}$$

$$\rho = \frac{1}{2}(re_1 + (r-2\cdot 1)e_2 + \cdots + (r-2\cdot r)e_{r+1})\,, \tag{B.3}$$

$$\zeta_h = \exp\left(\frac{2\pi i}{h}\right)\,, \tag{B.4}$$

$$\begin{aligned} \theta^2_{A_r}(\tau) &= q^{\frac{1}{24}\frac{r(r+2)}{h}} \prod_{n=1}^{\infty}(1-q^{\frac{n}{h}})^r \prod_{\alpha_k}\left(1 - q^{\frac{n}{h}}\zeta_h^{\langle\rho,\alpha_k\rangle}\right) \\ &= \frac{\eta(\tau)^{r+1}}{\eta(\tau/h)}, \end{aligned} \tag{B.5}$$

$$G_0(\tau) := \left(\frac{\theta^2_{A_r}(\tau)}{\eta(\tau)^{r+1}}\right)^{24} = \left(\frac{1}{\eta(\frac{\tau}{h})}\right)^{24}. \tag{B.6}$$

B.2 D_r case

$$\text{Roots}\{\alpha_k\}: \ \pm e_i \pm e_j\,, \quad (1 \le i \neq j \le r)\,. \tag{B.7}$$

$$h = 2r - 2\,, \tag{B.8}$$

$$\rho = \frac{1}{2}(he_1 + (h-2)e_2 + \cdots + 2e_{r-1})\,, \tag{B.9}$$

$$\zeta_h = \exp\left(\frac{2\pi i}{h}\right), \tag{B.10}$$

$$\begin{aligned} \theta^2_{D_r}(\tau) &= q^{\frac{1}{24}\frac{r(2r-1)}{h}} \prod_{n=1}^{\infty}(1-q^{\frac{n}{h}})^r \prod_{\alpha_k}\left(1 - q^{\frac{n}{h}}\zeta_h^{\langle\rho,\alpha_k\rangle}\right) \\ &= \frac{\eta(\tau)^{r+1}\eta(\frac{2\tau}{h})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{h})}, \end{aligned} \tag{B.11}$$

$$G_0(\tau) := \left(\frac{\theta^2_{D_r}(\tau)}{\eta(\tau)^{r+1}}\right)^{24} = \left(\frac{\eta(\frac{2\tau}{h})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{h})}\right)^{24}. \tag{B.12}$$

B.3 E_6 case

$$\text{Roots } \{\alpha_k\}:\ \pm e_i \pm e_j \quad (1 \le i \ne j \le 5)\,, \tag{B.13}$$

$$\frac{1}{2}(\pm e_1 \pm e_2 \cdots \pm \sqrt{3}e_6)\,, \qquad \text{even } +\text{'s}\,. \tag{B.14}$$

$$h = 12\,, \tag{B.15}$$

$$\rho = \frac{1}{2}(16e_1 + 6e_2 + 4e_3 + 2e_4)\,, \tag{B.16}$$

$$\zeta_{12} = \exp\left(\frac{2\pi i}{12}\right), \tag{B.17}$$

$$\begin{aligned} \theta^2_{E_6}(\tau) &= q^{\frac{1}{24}\frac{78}{12}} \prod_{n=1}^{\infty}(1-q^{\frac{n}{h}})^6 \prod_{\alpha_k}\left(1 - q^{\frac{n}{h}}\zeta_h^{\langle\rho,\alpha_k\rangle}\right) \\ &= \frac{\eta(\tau)^7\eta(\frac{\tau}{4})\eta(\frac{\tau}{6})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{12})}\,, \end{aligned} \tag{B.18}$$

$$G_0(\tau) := \left(\frac{\eta(\frac{\tau}{4})\eta(\frac{\tau}{6})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{12})}\right)^{24}. \tag{B.19}$$

B.4 E_7 case

$$\text{Roots } \{\alpha_k\}:\ \pm e_i \pm e_j \quad (1 \le i \ne j \le 6)\,, \quad \pm\sqrt{2}e_7\,, \tag{B.20}$$

$$\frac{1}{2}(\pm e_1 \pm e_2 \cdots \pm \sqrt{3}e_7)\,, \qquad \text{even } +\text{'s}\,. \tag{B.21}$$

$$h = 18\,, \tag{B.22}$$

$$\rho = \frac{1}{2}(26e_1 + 8e_2 + 6e_3 + 4e_4 + 2e_5 + \sqrt{2}e_7)\,, \tag{B.23}$$

$$\zeta_{18} = \exp\left(\frac{2\pi i}{18}\right), \tag{B.24}$$

$$\begin{aligned} \theta^2_{E_7}(\tau) &= q^{\frac{1}{24}\frac{133}{18}} \prod_{n=1}^{\infty}(1-q^{\frac{n}{h}})^7 \prod_{\alpha_k}\left(1 - q^{\frac{n}{h}}\zeta_h^{\langle\rho,\alpha_k\rangle}\right) \\ &= \frac{\eta(\tau)^8\eta(\frac{\tau}{6})\eta(\frac{\tau}{9})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{18})}\,, \end{aligned} \tag{B.25}$$

$$G_0(\tau) := \left(\frac{\eta(\frac{\tau}{6})\eta(\frac{\tau}{9})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{18})}\right)^{24} . \tag{B.26}$$

B.5 E_8 case

$$\text{Roots}\{\alpha_k\} : \ \pm e_i \pm e_j \quad (1 \le i \ne j \le 8) , \tag{B.27}$$

$$\frac{1}{2}(\pm e_1 \pm e_2 \cdots \pm e_8) , \qquad \text{even } +\text{'s} . \tag{B.28}$$

$$h = 30 , \tag{B.29}$$

$$\rho = \frac{1}{2}(46e_1 + 12e_2 + 10e_3 \cdots + 2e_7) , \tag{B.30}$$

$$\zeta_{30} = \exp\left(\frac{2\pi i}{30}\right) , \tag{B.31}$$

$$\begin{aligned} \theta^2_{E_8}(\tau) &= q^{\frac{1}{24}\frac{248}{30}} \prod_{n=1}^{\infty} (1 - q^{\frac{n}{h}})^8 \prod_{\alpha_k} \left(1 - q^{\frac{n}{h}} \zeta_h^{\langle \rho, \alpha_k \rangle}\right) \\ &= \frac{\eta(\tau)^9 \eta(\frac{\tau}{6})\eta(\frac{\tau}{10})\eta(\frac{\tau}{15})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{5})\eta(\frac{\tau}{30})} , \end{aligned} \tag{B.32}$$

$$G_0(\tau) := \left(\frac{\theta^2_{E_8}(\tau)}{\eta(\tau)^9}\right)^{24} = \left(\frac{\eta(\frac{\tau}{6})\eta(\frac{\tau}{10})\eta(\frac{\tau}{15})}{\eta(\frac{\tau}{2})\eta(\frac{\tau}{3})\eta(\frac{\tau}{5})\eta(\frac{\tau}{30})}\right)^{24} . \tag{B.33}$$

C. G functions of E_6 and E_7

C.1 Functions of E_6

We introduce G functions given by $\mathrm{SL}(2,\mathbb{Z})$ transformation from $G_0(\tau)$. Furthermore we add some G functions, so that all G functions satisfy the modular properties of SU(3).

$$G_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{4})\eta(\frac{\tau+j}{6})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})\eta(\frac{\tau+j}{12})}\right)^{24} , \qquad j = 0, \ldots, 11 , \tag{C.1}$$

$$H_j(\tau) := \left(\frac{\eta(\frac{2\tau+j}{2})\eta(\frac{2\tau+j}{3})}{\eta(2\tau)\eta(\frac{\tau-j}{3})\eta(\frac{2\tau+j}{6})}\right)^{24} , \qquad j = 0, \ldots, 6 , \tag{C.2}$$

$$I_j(\tau) := \left(\frac{\eta(\frac{\tau-j}{4})\eta(\frac{3\tau+j}{2})}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(\frac{3\tau+j}{4})}\right)^{24} , \qquad j = 0, \ldots, 3 , \tag{C.3}$$

$$J_j(\tau) := \left(\frac{\eta(4\tau)\eta(\frac{2\tau+j}{3})}{\eta(2\tau)\eta(\frac{\tau-j}{3})\eta(\frac{4\tau+2j}{3})}\right)^{24} , \qquad j = 0, \ldots, 2 , \tag{C.4}$$

$$K_j(\tau) := \left(\frac{\eta(\frac{2\tau+j}{2})\eta(6\tau)}{\eta(2\tau)\eta(3\tau)\eta(\frac{6\tau+j}{2})}\right)^{24} , \qquad j = 0, 1 , \tag{C.5}$$

$$L_0(\tau) := \left(\frac{\eta(4\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)\eta(12\tau)}\right)^{24} , \tag{C.6}$$

$$M_j(\tau) := \left(\frac{\eta(\tau)\eta(\frac{3\tau+3j}{2})}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(3\tau)}\right)^{24} , \qquad j = 0, 1 , \tag{C.7}$$

$$N_j(\tau) := \left(\frac{\eta(\tau)\eta(\frac{\tau+j}{6})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})\eta(\frac{\tau+j}{3})}\right)^{24}, \qquad j = 0, \ldots, 5. \tag{C.8}$$

To obtain $G_j(\tau), \ldots, L_0(\tau)$, we have to take account of the structure of order 12 Hecke transformation. $M_j(\tau)$ and $N_j(\tau)$ are obtained from $\mathrm{SL}(2,\mathbb{Z})$ transformation of these functions.

C.2 Function of E_7

$$G_j(\tau) := \left(\frac{\eta(\frac{\tau+j}{6})\eta(\frac{\tau+j}{9})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})\eta(\frac{\tau+j}{18})}\right)^{24}, \qquad j = 0, \ldots, 17, \tag{C.9}$$

$$H_j(\tau) := \left(\frac{\eta(\frac{2\tau+2j}{3})\eta(\frac{\tau+j}{9})}{\eta(2\tau)\eta(\frac{\tau+j}{3})\eta(\frac{2\tau+2j}{9})}\right)^{24}, \qquad j = 0, \ldots, 8, \tag{C.10}$$

$$I_j(\tau) := \left(\frac{\eta(\frac{3\tau+j}{2})\eta(\frac{3\tau+j}{3})}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(\frac{3\tau+j}{6})}\right)^{24}, \qquad j = 0, \ldots, 5, \tag{C.11}$$

$$J_j(\tau) := \left(\frac{\eta(6\tau)\eta(\frac{3\tau+j}{3})}{\eta(2\tau)\eta(3\tau)\eta(\frac{6\tau+2j}{3})}\right)^{24}, \qquad j = 0, \ldots, 2, \tag{C.12}$$

$$K_j(\tau) := \left(\frac{\eta(\frac{3\tau+j}{2})\eta(9\tau)}{\eta(\frac{\tau+j}{2})\eta(3\tau)\eta(\frac{9\tau+j}{2})}\right)^{24}, \qquad j = 0, 1, \tag{C.13}$$

$$L_0(\tau) := \left(\frac{\eta(6\tau)\eta(9\tau)}{\eta(2\tau)\eta(3\tau)\eta(18\tau)}\right)^{24}, \tag{C.14}$$

$$M_j(\tau) := \left(\frac{\eta(\tau)\eta(\frac{2\tau+2j}{3})}{\eta(2\tau)\eta(2\tau)\eta(\frac{\tau+j}{3})}\right)^{24}, \qquad j = 0, 1, 2, \tag{C.15}$$

$$N_j(\tau) := \left(\frac{\eta(\tau)\eta(\frac{\tau+j}{6})}{\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{2})\eta(\frac{\tau+j}{3})}\right)^{24}, \qquad j = 0, \ldots, 5. \tag{C.16}$$

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